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## Dynamic response of tower crane induced by the pendulum motion of the payload

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### Abstract

Dynamic response of tower cranes coupled with the pendulum motions of the payload is studied in this paper. A simple perturbation scheme and the assumption of small pendulum angle are applied to simplify the governing equation. The tower crane is modeled by the finite element method, while the pendulum motion is represented as rigid-body kinetics. Integrated governing equations for the coupled dynamics problem are derived based on Lagrange's equations including the dissipation function. Dynamics of a real luffing crane model with the spherical and planar pendulum motions is analyzed using the proposed formulations and computational method. It is found that the dynamic responses of the tower crane are dominated by both the first few natural frequencies of crane structure and the pendulum motion of the payload. The dynamic amplification factors generally increase with the increase of the initial pendulum angle and the changes are just slightly nonlinear for the planar pendulum motion.

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**Keywords:** Tower crane; Finite element; Pendulum; Structural dynamics

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### 1. Introduction

Tower cranes are used to lift and move heavy payloads in the construction of high-rise buildings. Under internal or external excitations, the payload always has a tendency to oscillate about its vertical position, resulting in a coupled dynamics problem of the vibration of the crane structure and the pendulum motion of the payload. This kind of pendulum-induced vibration may cause instability or serious damage to the

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crane system. Driven by the needs of crane design and control, efforts are made by researchers to understand the physical nature and engineering implication of the dynamics of crane systems including the payload. A recent review on crane dynamics, modeling and control is given by [Abdel-Rahman et al. \(2003\)](#).

Studies related to the dynamics and control of cranes have been mostly based on simplified models of crane structures. [Chin et al. \(2001\)](#) modeled a boom crane as a spherical pendulum and a rigid system with two degrees-of-freedom and assumed that the motion of the platform influences, but is not influenced by the swinging of the payload. The crane structure was also considered as rigid bodies with or without discrete springs in the studies carried by [Towarek \(1998\)](#), [Kiliceaslan et al. \(1999\)](#) and [Ghigliazza and Holmes \(2002\)](#). [Oguamanam et al. \(2001\)](#) studied the dynamics of overhead crane system where a beam model was used to represent the flexibility of crane structures. Generally speaking, these simplifications on the modeling of crane structures are reasonable when the dynamics of the pendulum motion of the payload is the main concern. However, when the stress and dynamic response of crane structures are of interest, a detailed modeling of crane structures is certainly needed. A tower crane is a complex structural system consisting of space frames, cable system and lumped mass and, therefore, finite element analysis appears to be an appropriate approach in this case. [Ju and Choo \(2002, 2005a\)](#) studied the natural vibration and dynamic response of the tower crane system due to the acceleration or deceleration of the payload, where detailed finite element modeling of crane system was employed. The motion of the payload in their study was confined to the vertical direction without any swing or pendulum motion.

The objective of this study is to derive integrated finite element formulations to analyze the dynamics of the tower crane coupled with the pendulum motion of the payload.

## 2. Theory and formulations

A real tower crane is a complex space structure including base, mast, jib, balance beam structures and machine system. [Fig. 1](#) schematically shows a simplified tower crane model and its deformed shape caused by the pendulum motion of the payload. As shown in the figure, the angle of the payload swing away from

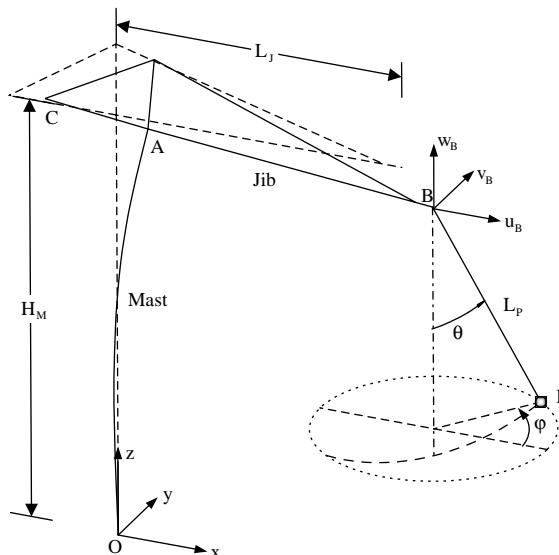


Fig. 1. Schematic geometry of deformed tower crane structure with payload pendulum.

the vertical line is denoted as  $\theta$ , while the angle of the payload rotating about the vertical line is denoted as  $\varphi$ . For the pure planar pendulum motion of the payload,  $\theta$  is a function of time but  $\varphi$  remains unchanged. For pure spherical pendulum motion of the payload,  $\varphi$  is a function of time and  $\theta$  is a constant. For the complex motion of the payload,  $\varphi$  and  $\theta$  are time dependent.

Critical parameters including the height of the mast ( $H_M$ ), length of the jib ( $L_J$ ) and the length of the pendulum ( $L_P$ ) are also indicated in Fig. 1. For the convenience of the kinetic analysis, the origin of the reference system is set at the base centre of the tower crane. The elastic displacements of the jib tip point B in the three directions, where the pendulum of the payload is connected to the jib structure, are denoted as  $u_B$ ,  $v_B$  and  $w_B$ , respectively.

## 2.1. Kinetic energy, potential energy and dissipation function

The position vector of payload  $P$  as shown in Fig. 1 can be expressed as

$$\mathbf{r}_P = (L_J + u_B + L_P \sin \theta \cos \varphi) \cdot \mathbf{i} + (v_B + L_P \sin \theta \sin \varphi) \cdot \mathbf{j} + (H_M + w_B - L_P \cos \theta) \cdot \mathbf{k} \quad (1)$$

where  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are unit vectors along the  $x$ -,  $y$ -, and  $z$ -axis, respectively. It can be seen that, by including the tip displacements ( $u_B$ ,  $v_B$  and  $w_B$ ), the effect of the elastic deformation of the crane structure is included in the position equation of the payload.

The velocity vector of the payload  $P$  can be obtained by the time derivative of  $\mathbf{r}_P$  as

$$\begin{aligned} \mathbf{v}_P = & (\dot{u}_B + L_P \cos \theta \cos \varphi \cdot \dot{\theta} - L_P \sin \theta \sin \varphi \cdot \dot{\varphi}) \cdot \mathbf{i} + (\dot{v}_B + L_P \cos \theta \sin \varphi \cdot \dot{\theta} + L_P \sin \theta \cos \varphi \cdot \dot{\varphi}) \cdot \mathbf{j} \\ & + (\dot{w}_B + L_P \sin \theta \cdot \dot{\theta}) \cdot \mathbf{k} \end{aligned} \quad (2)$$

Here the flexibility of the pendulum wire is neglected and  $L_P$  is therefore a constant.

The kinetic energy and the potential energy of the payload can then be derived as

$$\begin{aligned} T_P = & 1/2 m_P \mathbf{v}_P \cdot \mathbf{v}_P \\ = & 1/2 m_P \{ \dot{u}_B^2 + \dot{v}_B^2 + \dot{w}_B^2 + L_P^2 \dot{\theta}^2 + L_P^2 \sin^2 \theta \dot{\varphi}^2 + 2L_P \cos \theta \cos \varphi \dot{u}_B \dot{\theta} - 2L_P \sin \theta \sin \varphi \dot{u}_B \dot{\varphi} + 2L_P \\ & \times \cos \theta \sin \varphi \dot{v}_B \dot{\theta} + 2L_P \sin \theta \cos \varphi \dot{v}_B \dot{\varphi} + 2L_P \sin \theta \dot{w}_B \dot{\theta} \} \end{aligned} \quad (3a)$$

and

$$U_P = m_P g (H_M - L_P \cos \theta + w_B) \quad (3b)$$

where  $m_P$  and  $g$  are the mass of the payload and the acceleration of gravity, respectively.

Based on the finite element discretization, the kinetic and potential energies of tower crane structure can be respectively expressed as

$$T_C = \frac{1}{2} \{ \dot{\Delta} \}^T [\mathbf{M}] \{ \dot{\Delta} \} = \frac{1}{2} \left\{ \begin{array}{c} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{array} \right\}^T \left[ \begin{array}{cccc} \mathbf{M}_{rr} & \mathbf{M}_{ru} & \mathbf{M}_{rv} & \mathbf{M}_{rw} \\ \mathbf{M}_{ur} & m_{uu} & m_{uv} & m_{uw} \\ \mathbf{M}_{vr} & m_{vu} & m_{vv} & m_{vw} \\ \mathbf{M}_{wr} & m_{wu} & m_{wv} & m_{ww} \end{array} \right] \left\{ \begin{array}{c} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{array} \right\} \quad (4a)$$

and

$$U_C = \frac{1}{2} \{ \Delta \}^T [\mathbf{K}] \{ \Delta \} = \frac{1}{2} \left\{ \begin{array}{c} \Delta_r \\ u_B \\ v_B \\ w_B \end{array} \right\}^T \left[ \begin{array}{cccc} \mathbf{K}_{rr} & \mathbf{K}_{ru} & \mathbf{K}_{rv} & \mathbf{K}_{rw} \\ \mathbf{K}_{ur} & k_{uu} & k_{uv} & k_{uw} \\ \mathbf{K}_{vr} & k_{vu} & k_{vv} & k_{vw} \\ \mathbf{K}_{wr} & k_{wu} & k_{wv} & k_{ww} \end{array} \right] \left\{ \begin{array}{c} \Delta_r \\ u_B \\ v_B \\ w_B \end{array} \right\} \quad (4b)$$

Here  $[\mathbf{M}]$  and  $[\mathbf{K}]$  are the global mass and stiffness matrices of the tower crane structure;  $\Delta$  and  $\dot{\Delta}$  are displacement and velocity vectors of the whole system;  $(u_B, v_B, w_B)$  and  $(\dot{u}_B, \dot{v}_B, \dot{w}_B)$  are the nodal displacements and velocities of the jib tip point  $B$ , while  $\Delta_r$  and  $\dot{\Delta}_r$  are vectors of displacements and velocities for the rest of the degrees-of-freedom of the crane structure.

Similarly, the Rayleigh dissipation function can be expressed as

$$F_C = \frac{1}{2} \{ \dot{\Delta} \}^T [\mathbf{C}] \{ \dot{\Delta} \} = \frac{1}{2} \left\{ \begin{array}{c} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{array} \right\}^T \left[ \begin{array}{cccc} \mathbf{C}_{rr} & \mathbf{C}_{ru} & \mathbf{C}_{rv} & \mathbf{C}_{rw} \\ \mathbf{C}_{ur} & c_{uu} & c_{uv} & c_{uw} \\ \mathbf{C}_{vr} & c_{vu} & c_{vv} & c_{vw} \\ \mathbf{C}_{wr} & c_{wu} & c_{wv} & c_{ww} \end{array} \right] \left\{ \begin{array}{c} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{array} \right\} \quad (4c)$$

where  $[\mathbf{C}]$  is the damping matrix. Obviously,

$$\Delta = [(\Delta_r)^T \ u_B \ v_B \ w_B]^T; \quad \dot{\Delta} = [(\dot{\Delta}_r)^T \ \dot{u}_B \ \dot{v}_B \ \dot{w}_B]^T; \quad \ddot{\Delta} = [(\ddot{\Delta}_r)^T \ \ddot{u}_B \ \ddot{v}_B \ \ddot{w}_B]^T \quad (4d)$$

The total kinetic energy, potential energy and dissipation function of the crane system including crane structure and the pendulum of the payload are thus can be expressed as

$$\begin{aligned} T &= T_C + T_P \\ U &= U_C + U_P \\ F &= F_C \end{aligned} \quad (5)$$

where  $T_P$ ,  $U_P$ ,  $T_C$ ,  $U_C$  and  $F_C$  are given by Eqs. (3a,b)and (4a,b,c), respectively.

## 2.2. Integrated finite element formulations of governing equations

Based on Eqs. (5), the Lagrangian of the system can be expressed as

$$\begin{aligned} L = T - U &= \frac{1}{2} \left\{ \begin{array}{c} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{array} \right\}^T \left[ \begin{array}{cccc} \mathbf{M}_{rr} & \mathbf{M}_{ru} & \mathbf{M}_{rv} & \mathbf{M}_{rw} \\ \mathbf{M}_{ur} & m_{uu} & m_{uv} & m_{uw} \\ \mathbf{M}_{vr} & m_{vu} & m_{vv} & m_{vw} \\ \mathbf{M}_{wr} & m_{wu} & m_{wv} & m_{ww} \end{array} \right] \left\{ \begin{array}{c} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{array} \right\} \\ &+ \left( \begin{array}{l} \frac{1}{2} m_P \{ \dot{u}_B^2 + \dot{v}_B^2 + \dot{w}_B^2 + L_p^2 \dot{\theta}^2 + L_p^2 \sin^2 \theta \dot{\phi}^2 + \\ 2L_p \cos \theta \cos \varphi \dot{u}_B \dot{\theta} - 2L_p \sin \theta \sin \varphi \dot{u}_B \dot{\phi} + \\ 2L_p \cos \theta \sin \varphi \dot{v}_B \dot{\theta} + 2L_p \sin \theta \cos \varphi \dot{v}_B \dot{\phi} + \\ 2L_p \sin \theta \dot{w}_B \dot{\theta} \} \end{array} \right) \\ &- \frac{1}{2} \left\{ \begin{array}{c} \Delta_r \\ u_B \\ v_B \\ w_B \end{array} \right\}^T \left[ \begin{array}{cccc} \mathbf{K}_{rr} & \mathbf{K}_{ru} & \mathbf{K}_{rv} & \mathbf{K}_{rw} \\ \mathbf{K}_{ur} & k_{uu} & k_{uv} & k_{uw} \\ \mathbf{K}_{vr} & k_{vu} & k_{vv} & k_{vw} \\ \mathbf{K}_{wr} & k_{wu} & k_{wv} & k_{ww} \end{array} \right] \left\{ \begin{array}{c} \Delta_r \\ u_B \\ v_B \\ w_B \end{array} \right\} - m_P g (H_m - L_p \cos \theta + w_B) \end{aligned} \quad (6)$$

The Lagrange's equation including dissipation function is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} + \frac{\partial F}{\partial \dot{q}_r} = 0 \quad (7)$$

where  $q_r$  and  $\dot{q}_r$  are general coordinates and general velocities of the system. For the present problem,  $\Delta_r$ ,  $u_B$ ,  $v_B$ ,  $w_B$ ,  $\theta$  and  $\varphi$  are the general coordinates, and  $\dot{\Delta}_r$ ,  $\dot{u}_B$ ,  $\dot{v}_B$ ,  $\dot{w}_B$ ,  $\dot{\theta}$  and  $\dot{\varphi}$  are the corresponding general velocities.

Substituting Eqs. (4c) and (6) into Eq. (7) gives

$$\begin{aligned} & \left( [\mathbf{M}] + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & m_p & 0 & 0 \\ \mathbf{0} & 0 & m_p & 0 \\ \mathbf{0} & 0 & 0 & m_p \end{bmatrix} \right) \begin{Bmatrix} \ddot{\Delta}_r \\ \ddot{u}_B \\ \ddot{v}_B \\ \ddot{w}_B \end{Bmatrix} + [\mathbf{C}] \begin{Bmatrix} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} \Delta_r \\ u_B \\ v_B \\ w_B \end{Bmatrix} \\ &= \begin{Bmatrix} \mathbf{0} \\ m_p L_p (-\cos \theta \cos \varphi \ddot{\theta} + \sin \theta \cos \varphi \dot{\theta}^2 + 2 \cos \theta \sin \varphi \dot{\theta} \dot{\varphi} + \sin \theta \sin \varphi \ddot{\varphi} + \sin \theta \cos \varphi \dot{\varphi}^2) \\ m_p L_p (-\cos \theta \sin \varphi \ddot{\theta} + \sin \theta \sin \varphi \dot{\theta}^2 - 2 \cos \theta \cos \varphi \dot{\theta} \dot{\varphi} - \sin \theta \cos \varphi \ddot{\varphi} + \sin \theta \sin \varphi \dot{\varphi}^2) \\ -m_p L_p (\sin \theta \ddot{\theta} + \cos \theta \dot{\theta}^2) - m_p g \end{Bmatrix} \end{aligned} \quad (8a)$$

$$L_p \ddot{\theta} - L_p \sin \theta \cos \theta \dot{\varphi}^2 + \cos \theta \cos \varphi \ddot{u}_B + \cos \theta \sin \varphi \ddot{v}_B + \sin \theta \ddot{w}_B + g \sin \theta = 0 \quad (8b)$$

$$L_p \sin \theta \ddot{\varphi} + 2L_p \cos \theta \dot{\theta} \dot{\varphi} - \sin \varphi \ddot{u}_B + \cos \varphi \ddot{v}_B = 0 \quad (8c)$$

Eqs. (8a)–(8c) are the integrated finite element formulations for structural dynamics of the tower crane coupled with the pendulum motion of the payload.

### 2.3. Simplification for small pendulum angle

For small angular displacement,  $\sin \theta$  can be replaced by  $\theta$  and  $\cos \theta$  by 1 (Nelson and Olsson, 1986), and Eqs. (8a)–(8c) are then simplified to

$$\begin{aligned} & \left( [\mathbf{M}] + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & m_p & 0 & 0 \\ \mathbf{0} & 0 & m_p & 0 \\ \mathbf{0} & 0 & 0 & m_p \end{bmatrix} \right) \begin{Bmatrix} \ddot{\Delta}_r \\ \ddot{u}_B \\ \ddot{v}_B \\ \ddot{w}_B \end{Bmatrix} + [\mathbf{C}] \begin{Bmatrix} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} \Delta_r \\ u_B \\ v_B \\ w_B \end{Bmatrix} \\ &= \begin{Bmatrix} \mathbf{0} \\ m_p L_p (-\cos \varphi \ddot{\theta} + \theta \cos \varphi \dot{\theta}^2 + 2 \sin \varphi \dot{\theta} \dot{\varphi} + \theta \sin \varphi \ddot{\varphi} + \theta \cos \varphi \dot{\varphi}^2) \\ m_p L_p (-\sin \varphi \ddot{\theta} + \theta \sin \varphi \dot{\theta}^2 - 2 \cos \varphi \dot{\theta} \dot{\varphi} - \theta \cos \varphi \ddot{\varphi} + \theta \sin \varphi \dot{\varphi}^2) \\ -m_p L_p (\theta \ddot{\theta} + \dot{\theta}^2) - m_p g \end{Bmatrix} \end{aligned} \quad (9a)$$

$$L_p \ddot{\theta} - L_p \theta \dot{\varphi}^2 + \cos \varphi \ddot{u}_B + \sin \varphi \ddot{v}_B + \theta \ddot{w}_B + g \theta = 0 \quad (9b)$$

$$L_p \theta \ddot{\varphi} + 2L_p \dot{\theta} \dot{\varphi} - \sin \varphi \ddot{u}_B + \cos \varphi \ddot{v}_B = 0 \quad (9c)$$

It can be seen that, even with the assumption of small pendulum angle, the simplified differential equations, Eqs. (9a)–(9c) are still nonlinear for the coupled structural dynamics with the pendulum motion.

### 2.4. A special case with rigid structure assumption

If the crane structure is assumed to be rigid, Eqs. (8a)–(8c) will degenerate to the following nonlinear differential equations

$$L_p \ddot{\theta} - L_p \theta \dot{\varphi}^2 + g \theta = 0 \quad (10a)$$

$$L_p \theta \ddot{\varphi} + 2L_p \dot{\theta} \dot{\varphi} = 0 \quad (10b)$$

Eqs. (10a) and (10b) can also be obtained directly from Newton's Second Law of Motion.

Furthermore, if the movement of payload is confined to a vertical plane with  $\varphi = \varphi_0$ ,  $\dot{\varphi} = \ddot{\varphi} = 0$  and  $\theta|_{t=0} = \theta_0$ , Eq. (10a) reduces to

$$L_P \ddot{\theta} + g\theta = 0. \quad (11a)$$

Eq. (11a) is the well known equation of the linear planar pendulum (Nelson and Olsson, 1986) and its solution is

$$\theta = \theta_0 \cos(\omega_0 t) \quad (11b)$$

where  $\omega_0$  is the natural frequency of the pendulum given by

$$\omega_0 = \sqrt{g/L_P} \quad (11c)$$

Similarly, if the spherical motion of the payload is assumed (Ghigliazza and Holmes, 2002; Markeyev, 1998), with  $\theta = \theta_0$ ,  $\dot{\theta} = \ddot{\theta} = 0$  and  $\varphi|_{t=0} = \varphi_0$ , the equation governing the motion of the spherical pendulum of the payload is obtained as

$$-L_P \dot{\varphi}^2 + g = 0 \quad (\text{with } \ddot{\varphi} = 0) \quad (12a)$$

And the solution of Eq. (12a) is

$$\varphi = \omega_0 t + \varphi_0 \quad (12b)$$

where  $\omega_0$  is given in Eq. (11c). It can be seen that, with the rigid structure assumption, the spherical pendulum of the payload is a uniform rotation with the angular speed equal to the natural frequency of the corresponding linear planar pendulum.

### 3. Approximation for numerical computation

Solving the coupled equations for structural dynamics and the pendulum motion may involve the complex nonlinear dynamical phenomena of bifurcation and chaos (Chin et al., 2001; Schwartz et al., 1999; Katriel, 2002). Two special cases of the motions of the payload, namely spherical pendulum and planar pendulum as discussed in the previous sections, are considered in this paper to study the coupled structural dynamics of tower cranes.

#### 3.1. Flexible crane structure with spherical pendulum

Eq. (12b) gives the solution of the spherical pendulum of the payload based on the assumption of rigid structure, and this solution is not valid for the flexible crane structure. Based on the premise that the elastic deformation of the crane structure is small compared to the pendulum motion of the payload, it is assumed in this study that the approximated solution for the spherical pendulum with flexible structure is of the form

$$\varphi = \omega_0 t + \varphi_0 + \varepsilon_\varphi(t) \quad (13a)$$

where  $\varepsilon_\varphi(t)$  is small perturbation. It can be seen that an additional term,  $\varepsilon_\varphi(t)$ , is added to Eq. (12b) to get Eq. (13a).  $\varepsilon_\varphi(t)$  is employed to account for the flexibility of the structure.

The angular velocity and acceleration, derived from Eq. (13a), are

$$\begin{aligned} \dot{\varphi} &= \omega_0 + \dot{\varepsilon}_\varphi(t) \\ \ddot{\varphi} &= \ddot{\varepsilon}_\varphi(t) \end{aligned} \quad (13b)$$

Here, like  $\varepsilon_\varphi(t)$ ,  $\dot{\varepsilon}_\varphi(t)$  and  $\ddot{\varepsilon}_\varphi(t)$  are also assumed to be small perturbation terms.

Substituting Eqs. (13a) and (13b) into Eqs. (9a)–(9c) gives

$$\begin{aligned} & \left( [\mathbf{M}] + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & m_p & 0 & 0 \\ \mathbf{0} & 0 & m_p & 0 \\ \mathbf{0} & 0 & 0 & m_p \end{bmatrix} \right) \begin{Bmatrix} \ddot{\Delta}_r \\ \ddot{u}_B \\ \ddot{v}_B \\ \ddot{w}_B \end{Bmatrix} + [\mathbf{C}] \begin{Bmatrix} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} \Delta_r \\ u_B \\ v_B \\ w_B \end{Bmatrix} \\ &= \begin{Bmatrix} \mathbf{0} \\ m_p[\ddot{u}_B + \theta_0 \cos(\omega_0 t + \varphi_0) \ddot{w}_B] + m_p \theta_0 g \cos(\omega_0 t + \varphi_0) \\ m_p[\ddot{v}_B + \theta_0 \sin(\omega_0 t + \varphi_0) \ddot{w}_B] + m_p \theta_0 g \sin(\omega_0 t + \varphi_0) \\ -m_p g \end{Bmatrix} \end{aligned} \quad (14a)$$

$$2\omega_0 \theta_0 \dot{\varepsilon}_\varphi = \theta_0 \ddot{w}_B + \cos(\omega_0 t + \varphi_0) \ddot{u}_B + \sin(\omega_0 t + \varphi_0) \ddot{v}_B \quad (14b)$$

$$L_p \theta_0 \ddot{\varepsilon}_\varphi = \sin(\omega_0 t + \varphi_0) \ddot{u}_B - \cos(\omega_0 t + \varphi_0) \ddot{v}_B \quad (14c)$$

where all product terms of  $\varepsilon_\varphi(t)$ ,  $\dot{\varepsilon}_\varphi(t)$  and  $\ddot{\varepsilon}_\varphi(t)$  are dropped.

Eq. (14a) can be further simplified to:

$$[\mathbf{M}] \begin{Bmatrix} \ddot{\Delta}_r \\ \ddot{u}_B \\ \ddot{v}_B \\ \ddot{w}_B \end{Bmatrix} + [\mathbf{C}] \begin{Bmatrix} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} \Delta_r \\ u_B \\ v_B \\ w_B \end{Bmatrix} = m_p(\ddot{w}_B + g) \theta_0 \begin{Bmatrix} \mathbf{0} \\ \cos(\omega_0 t + \varphi_0) \\ \sin(\omega_0 t + \varphi_0) \\ -1 \end{Bmatrix} \quad (14d)$$

It is interesting to notice that the governing equation for the dynamic response of the tower crane is decoupled from the equation of the spherical pendulum motion of the payload. This phenomenon was also reported in Oguamanam et al. (2001). The right-hand side of Eq. (14d) indicates that the dynamic response of the crane structure is proportional to the amplitude of the pendulum angle  $\theta_0$ , regardless of the length of the pendulum when the payload experiences the motion of the spherical pendulum.

### 3.2. Flexible crane structure with planar pendulum

Similarly, based on Eq. (11b), the solution for the planar pendulum with flexible structure is assumed to be the following form

$$\theta = \theta_0 \cos(\omega_0 t) + \varepsilon_\theta(t) \quad (15a)$$

where  $\varepsilon_\theta(t)$  is a small perturbation term. The angular velocity and acceleration are

$$\begin{aligned} \dot{\theta} &= -\omega_0 \theta_0 \sin(\omega_0 t) + \dot{\varepsilon}_\theta(t) \\ \ddot{\theta} &= -\omega_0^2 \theta_0 \cos(\omega_0 t) + \ddot{\varepsilon}_\theta(t) \end{aligned} \quad (15b)$$

Here  $\dot{\varepsilon}_\theta(t)$  and  $\ddot{\varepsilon}_\theta(t)$  are also small perturbations.

Substituting Eqs. (15a) and (15b) into Eqs. (9a)–(9c) yields

$$\left( [\mathbf{M}] + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & m_p & 0 & 0 \\ \mathbf{0} & 0 & m_p & 0 \\ \mathbf{0} & 0 & 0 & m_p \end{bmatrix} \right) \begin{Bmatrix} \ddot{\Delta}_r \\ \ddot{u}_B \\ \ddot{v}_B \\ \ddot{w}_B \end{Bmatrix} + [\mathbf{C}] \begin{Bmatrix} \dot{\Delta}_r \\ \dot{u}_B \\ \dot{v}_B \\ \dot{w}_B \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} \Delta_r \\ u_B \\ v_B \\ w_B \end{Bmatrix}$$

$$= \begin{cases} \mathbf{0} \\ m_P \cos \varphi_0 [-L_P \ddot{\theta} - 2\omega_0 \theta_0^2 L_P \cos(\omega_0 t) \sin(\omega_0 t) \dot{\theta} + 2g\theta_0^2 \sin^2(\omega_0 t) \cos \varphi_0 \dot{\varepsilon}_\theta \\ \quad + g\theta_0 \cos(\omega_0 t) + g\theta_0^3 \sin^2(\omega_0 t) \cos(\omega_0 t)] \\ m_P \sin \varphi_0 [-L_P \ddot{\theta} - 2\omega_0 \theta_0^2 L_P \cos(\omega_0 t) \sin(\omega_0 t) \dot{\theta} + 2g\theta_0^2 \sin^2(\omega_0 t) \cos \varphi_0 \dot{\varepsilon}_\theta \\ \quad + g\theta_0 \cos(\omega_0 t) + g\theta_0^3 \sin^2(\omega_0 t) \cos(\omega_0 t)] \\ m_P [-\theta_0 L_P \cos(\omega_0 t) \ddot{\theta} + 2\omega_0 \theta_0 L_P \sin(\omega_0 t) \dot{\theta} + g\theta_0 \cos^2(\omega_0 t) \dot{\varepsilon}_\theta \\ \quad + 2g\theta_0^2 \cos(2\omega_0 t)] - m_P g \end{cases} \quad (16a)$$

$$L_P \ddot{\theta} + (\ddot{w}_B + g) \dot{\varepsilon}_\theta = -\theta_0 \ddot{w}_B \cos(\omega_0 t) - \ddot{u}_B \cos \varphi_0 - \ddot{v}_B \sin \varphi_0 \quad (16b)$$

$$0 = \ddot{u}_B \sin \varphi_0 - \ddot{v}_B \cos \varphi_0 \quad (16c)$$

It can be seen from Eqs. (16a)–(16c) that the dynamic response of tower crane is fully coupled with the planar pendulum motion of the payload.

Eq. (16c) serves as a constraint for the tip acceleration and  $\ddot{u}_B = \ddot{v}_B \cdot \cos \varphi_0 / \sin \varphi_0$  (for  $\sin \varphi_0 \neq 0$ ) or  $\ddot{v}_B = \ddot{u}_B \cdot \sin \varphi_0 / \cos \varphi_0$  (for  $\cos \varphi_0 \neq 0$ ) will be substituted into Eq. (16b) to impose such constraint.

The computational scheme for solving Eqs. (16a)–(16c) is based on Newmark method (Bathe and Wilson, 1976; Nickel, 1971) and iterative approach (Kelley, 1995; Das and Sargand, 1999). The computational procedures with a time interval of  $\Delta t$  can be summarized as:

- Step 1. Given  $\Delta^{(t)}$ ,  $\dot{\Delta}^{(t)}$  and  $\ddot{\Delta}^{(t)}$ , get  $\Delta^{(t + \Delta t)}$ ,  $\dot{\Delta}^{(t + \Delta t)}$  and  $\ddot{\Delta}^{(t + \Delta t)}$  from Eq. (16a) while neglecting  $\varepsilon_\theta$ ,  $\dot{\varepsilon}_\theta$  and  $\ddot{\varepsilon}_\theta$ ;
- Step 2. Substitute  $\ddot{u}_B^{(t + \Delta t)}$ ,  $\ddot{v}_B^{(t + \Delta t)}$  and  $\ddot{w}_B^{(t + \Delta t)}$  into Eq. (16b), get  $\dot{\varepsilon}_\theta^{(t + \Delta t)}$ ,  $\ddot{\varepsilon}_\theta^{(t + \Delta t)}$  and  $\ddot{\varepsilon}_\theta^{(t + \Delta t)}$ ;
- Step 3. Substitute  $\dot{\varepsilon}_\theta^{(t + \Delta t)}$ ,  $\ddot{\varepsilon}_\theta^{(t + \Delta t)}$  and  $\ddot{\varepsilon}_\theta^{(t + \Delta t)}$  into Eq. (16a), get new  $\Delta^{(t + \Delta t)}$ ,  $\dot{\Delta}^{(t + \Delta t)}$  and  $\ddot{\Delta}^{(t + \Delta t)}$ ;
- Step 4. Check the convergence of  $\ddot{u}_B^{(t + \Delta t)}$ ,  $\ddot{v}_B^{(t + \Delta t)}$  and  $\ddot{w}_B^{(t + \Delta t)}$ , go back to Step 2 if convergence condition is not satisfied.

#### 4. Numerical results and discussions

A real model of Potain luffing tower crane is used for numerical analysis in this study, where the total height of the crane is 68.3 metres. The types of the finite elements used in modeling the crane structures are mostly space frame and truss elements. The counter weight, equipment and the payload are modeled by lump mass and the luffing and hoisting cables are modeled by cable-pulley element proposed by Ju and Choo (2005b).

##### 4.1. Natural modes and frequencies of the tower crane structure

To study the effect of pendulum parameters on the coupled vibration of the tower cranes, the natural vibration of the tower crane structures without the payload is firstly analyzed. Fig. 2 gives the first four natural mode and frequencies of the tower crane. It can be seen that the first mode is dominated by the deformation of the jib structure, while the second and the third modes are predominantly the complex bending patterns of the whole crane structure. Twist of the jib structure is found in the fourth mode, coupled with the bending of the mast and the jib. Detailed discussion on the natural vibration of crane structures can be found in the study by Ju and Choo (2002).

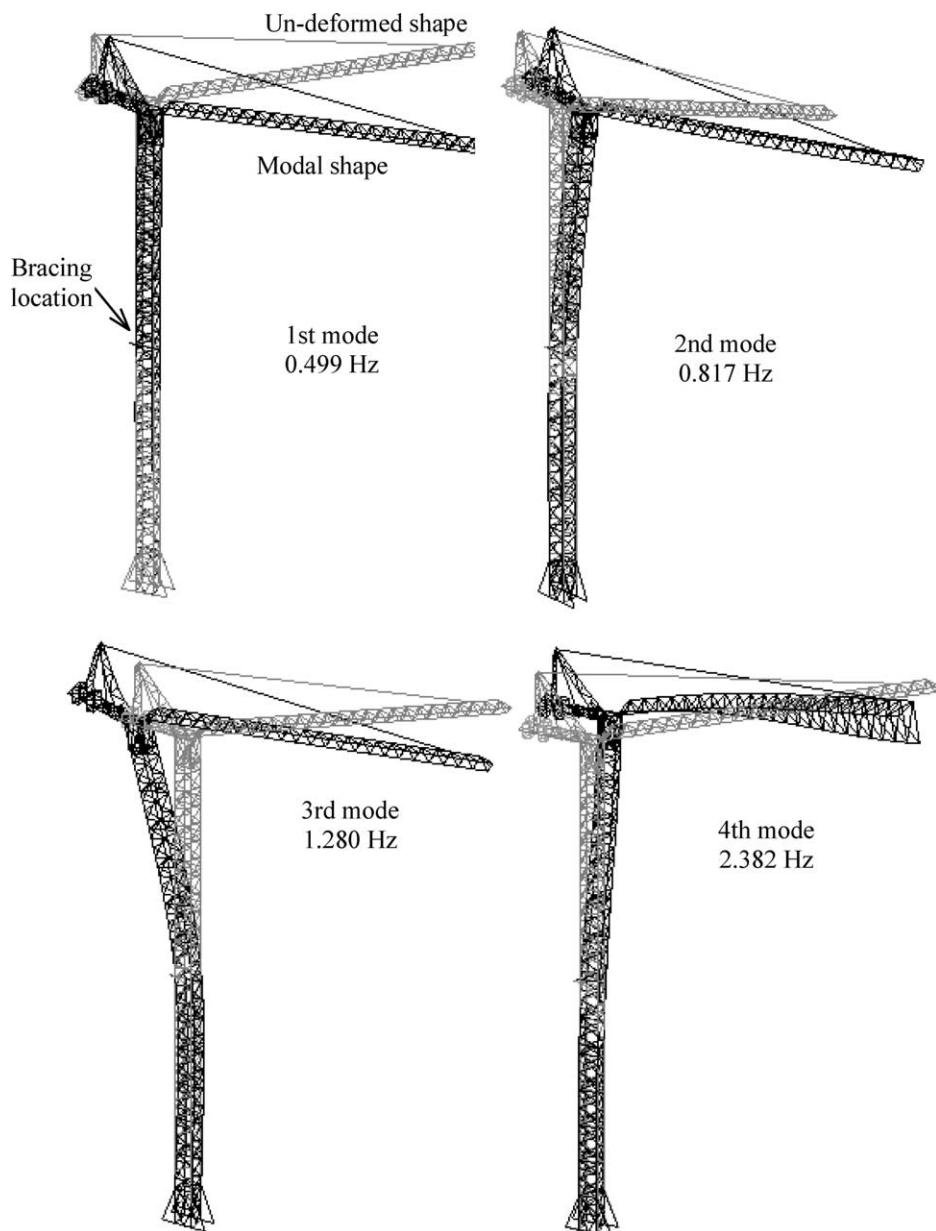


Fig. 2. First four natural modes of a tower crane structure with bracings at the mast.

#### 4.2. Dynamic response of tower crane with the spherical pendulum motion of the payload

Fig. 3 shows the deformed shape of a tower crane, where the payload is experiencing the spherical pendulum motion with an initial pendulum angle,  $\theta_0$ , of  $10.0^\circ$ . The length of the pendulum,  $L_P$ , is 40 m, while the weight of the payload is 2000 kg.

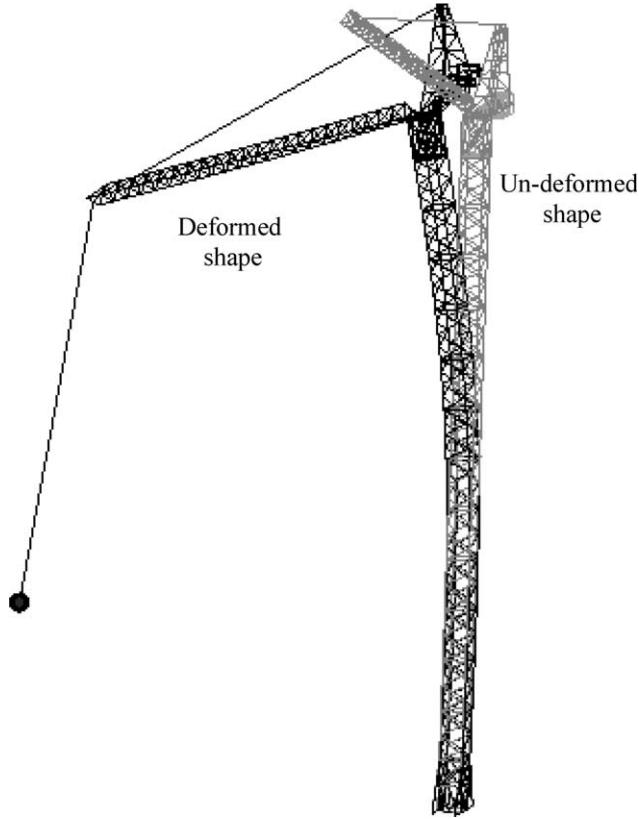


Fig. 3. Deformed shape of crane structure with the spherical pendulum motion of the payload ( $\theta_0 = 10.0^\circ$ ).

Fig. 4a shows the factored dynamic response of the tip node B,  $v_B(t)/(w_B)_0$  and  $w_B(t)/(w_B)_0$ . Here  $v_B(t)$  and  $w_B(t)$  are the dynamic responses of displacements of the node B in y and z directions;  $(w_B)_0$  is the static displacement of the node B at vertical direction without any motion of the payload. The reason for using  $v_B(t)/(w_B)_0$  instead of  $v_B(t)/(v_B)_0$  is that  $(v_B)_0$  is very small when only a vertical load is acting at the tip node. Fig. 4b gives the spectrum of the factored power density of  $w_B(t)/(w_B)_0$  shown in Fig. 4a. It can be seen from the figure that there are mainly three peak points at the power density spectrum, located, respectively, at 0.08 Hz, 0.50 Hz and 0.80 Hz. The last two frequencies clearly correspond to the first and second natural modes given in Fig. 2, while 0.08 Hz is the natural frequency of the pendulum given by  $\omega_0 = \frac{1}{2\pi} \sqrt{g/L_p} (\approx 0.079 \text{ Hz})$ . The dynamic response of crane structure is therefore dominated by these three frequencies in this case.

#### 4.3. Dynamic response of tower crane with the planar pendulum motion of the payload

The factored dynamic response of the tip node B,  $v_B(t)/(w_B)_0$  and  $w_B(t)/(w_B)_0$ , for the planar pendulum motion of the payload is also computed based on Eqs. (16a)–(16c) and shown in Fig. 5a, while the spectrum of power density of  $w_B(t)/(w_B)_0$  is shown in Fig. 5b.

The dynamic response of tower crane with the planar pendulum motion of the payload is more complex than that with the spherical pendulum motion. The power density spectrum in Fig. 5b shows more peaks for the planar pendulum motion than spherical pendulum motion. The last three peak frequencies, 0.50 Hz,

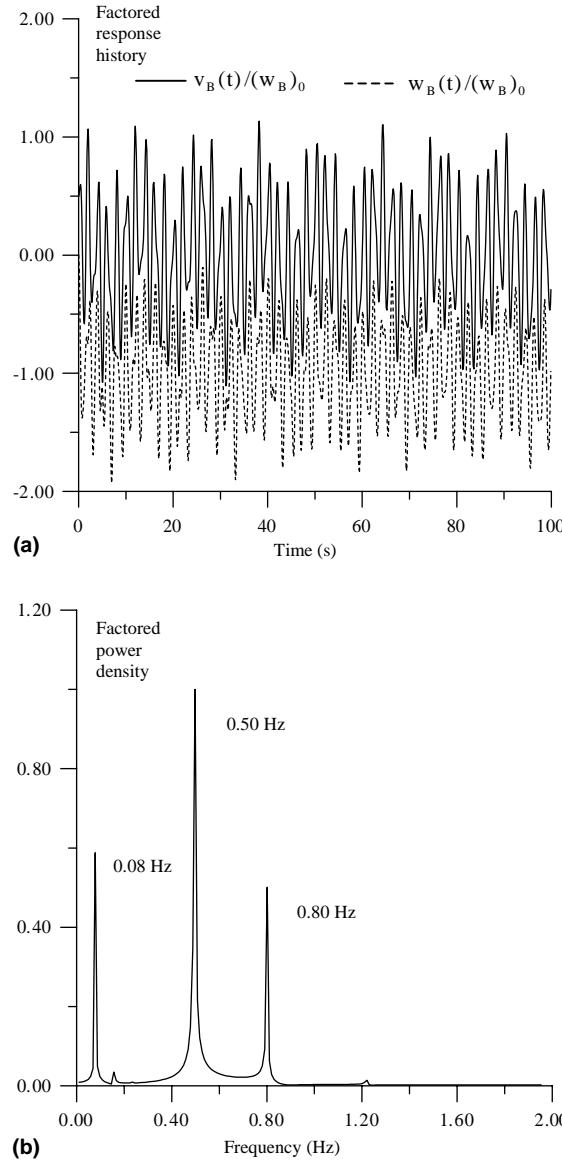


Fig. 4. (a) Displacement response of the tip point B ( $w_B$ ) coupled with spherical pendulum motion of the payload ( $\theta_0 = 2.0^\circ$ ) and (b) factored power density of the vertical response of tip point ( $w_B$ ) with the spherical pendulum motion of the payload.

0.80 Hz and 1.22 Hz correspond to the first three natural frequencies of the tower crane given in Fig. 2. The first peak frequency, 0.16 Hz, is the second harmonic of the natural frequency of the pendulum  $\omega_0$  due to the presence of  $\cos(2\omega_0 t)$  in the right-hand side of Eq. (16a).

Fig. 6 shows the effect of the initial pendulum angle,  $\theta_0$ , on the dynamic amplification factor of the displacement response of the tip point, which is defined as the maximum amplitude of displacement response over the static displacement. It is found that the dynamic amplification factors increase with the increase of the initial pendulum angle and the changes are slightly nonlinear. It is not surprising to see that the

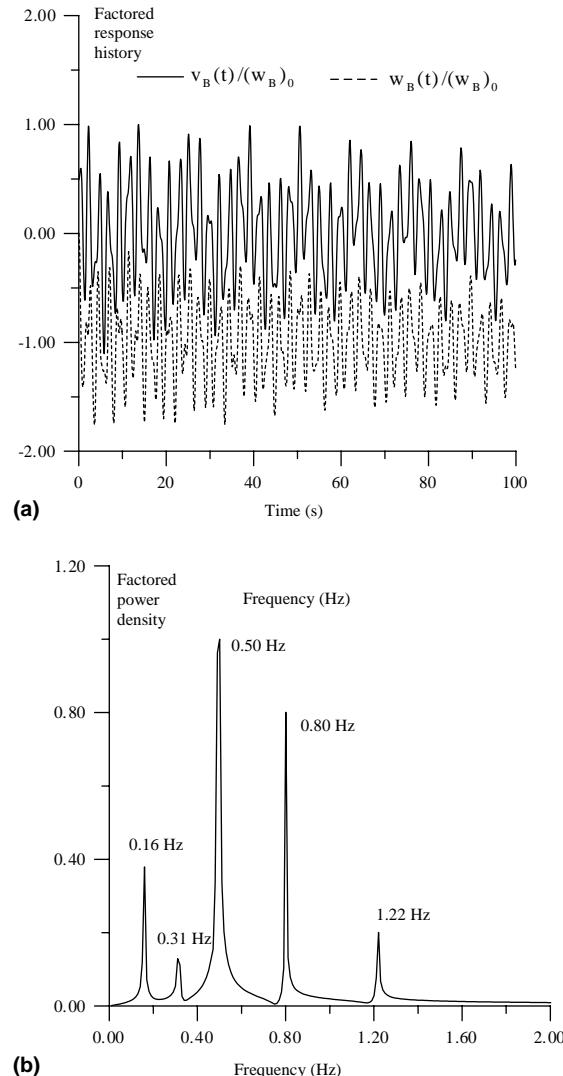


Fig. 5. (a) Displacement response of the tip point B coupled with planar pendulum motion of the payload ( $\theta_0 = 2.0^\circ$ ) and (b) factored power density of the vertical response of tip point  $(w_B)$  with the planar pendulum motion of the payload.

maximum response at lateral direction (y direction as shown in Fig. 1),  $(v_B)_{\max}$ , is greater than that at vertical direction,  $(w_B)_{\max}$ , since the tower crane structure is weaker at that direction.

## 5. Conclusions

Governing equations for the dynamic response of a tower cranes coupled with the pendulum motion of the payload are derived based on Lagrange's equations including the dissipation function. The tower crane is modeled based on finite element method, while the pendulum motion is represented as a multi-body system. The derived equations are then simplified by the assumption of small pendulum angle, resulting in a set

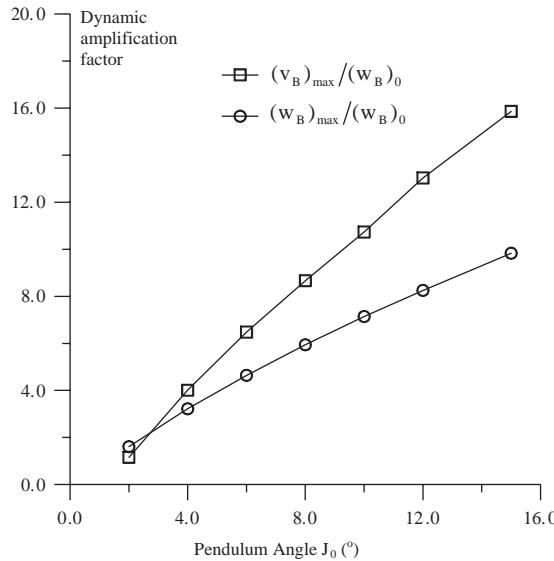


Fig. 6. Dynamic amplification factor of tip displacement with the planar pendulum motion of the payload.

of coupled differential equations with nonlinear excitation loads. These equations are essentially representing the coupled engineering problem of structural dynamics and multi-body dynamics are difficult to solve analytically or numerically.

If the crane structure is assumed to be rigid, the derived governing equations correctly degenerate to nonlinear differential equations, which exactly satisfy Newton's Law of Motion for the spherical and planar pendulum motions. It is found the natural period of spherical pendulum equals to that of planar pendulum with the same length.

Based on the exact solution of pure spherical and planar pendulums, a simple first-order perturbation approach is used to further simplify the governing equations for dynamics of tower cranes with flexible structures and the pendulum motion of the payload. This perturbation-based simplification gives a set of solvable coupled equations, which are then computed based on Newmark method with iterative schemes.

Numerical studies are then carried out for a real luffing tower crane with both the spherical and the planar pendulum motions of the payload. It is found the dynamic responses of the tower crane are dominated by the first two natural frequencies of crane structure and the natural frequency of the pendulum for the spherical pendulum motion. For the planar pendulum motion of the payload, the dynamic response is mainly contributed by the first three natural modes and a second harmonic of the pendulum. It is also found that the dynamic amplification factors increase with the increase of the initial pendulum angle and the changes are just slightly nonlinear for the planar pendulum motion of the payload.

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